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HYPERCONTRACTIVE SEMIGROUPS AND SOBOLEV'S INEQUALITY

BY

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ABSTRACT. If $H \geqslant 0$ is the generator of a hypercontractive semigroup (HCSG), it is known that $(H + 1)^{-1/2}$ is a bounded operator from L^p to L^p, $1 \leqslant p \leqslant \infty$. We prove that $(H + 1)^{-1/2}$ is bounded from L^2 to the Orlicz space $L^2 \ln^+ L$, basing the proof on the uniform semiboundedness of the operator $H + V$, for suitable V. We also prove by an interpolation argument, that $(H + 1)^{-1/2}$ is bounded from L^p to $L^p \ln^+ L$, $2 \leqslant p < \infty$. Another interpolation argument shows that $(H + 1)^{-1/2}$ is bounded from $L^p(\ln^+ L)^m$ to $L^p(\ln^+ L)^{m+1}$, $2 \leqslant p < \infty$ and m a positive integer. Finally, we identify the topological duals of the spaces mentioned above.

1. Introduction. The classical Sobolev inequality states (in part) that a real-valued, square-integrable function on R^n whose first partial derivatives are also square-integrable (with respect to Lebesgue measure) is pth-power integrable, where $1/p = 1/2 - 1/n$ [7].

In attempting to extend this result to an infinite-dimensional measure space, two problems immediately arise. First, as $n \longrightarrow \infty$, $p \longrightarrow 2$, so less and less information is gained. Secondly, and more importantly, there is no infinite-dimensional analogue to Lebesgue measure. To avoid this second problem, we seek a version of the inequality that holds on a probability space, such as $(R, \pi^{-1/2} \exp(-x^2)dx)$, that is, the real line with Gaussian measure. Then, the extension of the result to an arbitrary product of such spaces is relatively direct.

Using this example, it is easy to see that no analogue of Sobolev's inequality involving only Lebesgue spaces exists. Indeed, given $\epsilon > 0$, there is a function f such that both f and f' are in $L^2(R, \pi^{-1/2}\exp(-x^2)\,dx)$, but f is not in $L^{2+\epsilon}$. However, a version involving Orlicz spaces does exist, namely, if f and f' are in $L^2(R, \pi^{-1/2}\exp(-x^2)\,dx)$, then not only is $|f|^2$ integrable, but so is $|f|^2\ln |f|$ [3].

This result, as well as the extensions in §4 appeared in the author's dissertation [3].

As will be seen, this result is intimately connected with the theory of hypercontractive semigroups [11]. We refer the readers to *Logarithmic Sobolev ineqal-*

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ities by L. Gross [4] where further information concerning this relationship is obtained.

2. **Hypercontractive semigroups.**

2.1 DEFINITION. Let (M, μ) be a probability space and let $H \geqslant 0$ be a self-adjoint operator (not necessarily bounded) on $L^2(M, d\mu)$. Then $\{\exp(-tH)\}_{t\geqslant 0}$ is a hypercontractive semigroup (HCSG) iff:

(i) $\exp(-tH)$: $L^1(M, d\mu) \longrightarrow L^1(M, d\mu)$ is a contraction for all $t > 0$.

(ii) For some $T > 0$, $\exp(-TH)$ is bounded from $L^2(M, d\mu)$ to $L^4(M, d\mu)$.

This T is denoted $T(H)$ whenever confusion could occur [11].

In this paper we assume that $\exp(-tH)$ is also positivity preserving.

2.2 PROPOSITION. *Let* $\exp(-tH)$ *be a HCSG. Then*:

(i) $\exp(-tH)$ *is a contraction on* $L^p(M, d\mu)$, $1 \leqslant p \leqslant \infty$, *for all* $t > 0$.

(ii) *Given* $1 < a < b < \infty$, *there are positive constants* $T_{a,b}$ *and* C *such that*

$$\|\exp(-tH)\psi\|_p \leqslant C\|\psi\|_q$$

for all $\psi \in L^q$, *for all* $t > T_{a,b}$, *and all* p, q *satisfying* $a < p, q < b$.

PROOF. See [11].

The following theorem of Segal ([10] and [11]) provides information on the behavior of $\exp(-tH)$ for t near 0.

2.3 PROPOSITION. *Let* $\{\exp(-tH)\}_{t\geqslant 0}$ *be a HCSG. For* $\lambda > 0$, *and* $t > 0$, *define* $p(t, \lambda)^{-1} = 1/2 - t\lambda$. *Then there exist positive constants* C *and* λ *depending only on* H *such that*

$$\|\exp(-tH)\psi\|_{p(t,\lambda)} \leqslant C^t\|\psi\|_2$$

for all $\psi \in L^2$ *and all* $t < T(H)$.

PROOF. Consider the function $f(z) = \exp(-zH)$ defined in $0 \leqslant \operatorname{Re}(z) \leqslant T$. f is continuous in the closed strip and analytic in its interior. If $\psi \in L^2$ and $\operatorname{Re}(z) = 0$, $\|f(z)\psi\|_2 \leqslant \|\psi\|_2$ by Proposition 2.2. If $\operatorname{Re}(z) = T$, then $\|f(z)\psi\|_4 \leqslant C\|\psi\|_2$ where C is the bound on $\exp(-tH)$ as an operator from L^2 to L^4. It follows then from the Stein interpolation theorem [12] that for $0 < t < T$, $\|f(t)\psi\|_p \leqslant A_t\|\psi\|_2$ where $1/p = (1 - t/T)/2 + (t/T)/4$ and $A_t = C^t$. Setting $\lambda = 1/(4T)$ gives the desired result.

In the following, we assume w.l.o.g. that $C > 1$.

Another theorem from Simon and Hoegh-Krohn's paper that we will use is:

2.4 PROPOSITION. *Let* H *be the generator of a HCSG. Then there exist real* *numbers, S, E, and m, dependent only on H, such that for all V in L^∞,*

$$(*)\qquad \|\exp(-S(H+V))\psi\|_2 \leqslant E\|\exp(-SV)\|_m \|\psi\|_2.$$

PROOF. See [11].

2.5 PROPOSITION. *Let H be the generator of a HCSG. Then the operator $(H+1)^{-1/2}$ is bounded from L^∞ to L^∞.*

PROOF. It will be sufficient to show that $(H+1)^{-1/2}$ is bounded as an operator from L^1 to L^1 and use the fact that it is selfadjoint. Consider the weak integral

$$(H+1)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-t(H+1))t^{-1/2}\,dt.$$

Let $u \in L^1$. Since $t \longrightarrow \exp(-tH)u$ is continuous from R to L^1,

$$\begin{aligned}\|(H+1)^{-1/2}u\|_1 &= \frac{1}{\sqrt{\pi}} \int_M \left| \int_0^\infty \exp(-t(H+1))ut^{-1/2}\,dt \right| d\mu \\ &\leqslant \frac{1}{\sqrt{\pi}} \int_0^\infty \int_M |\exp(-t(H+1))u|t^{-1/2}\,dt\,d\mu \\ &\leqslant \frac{1}{\sqrt{\pi}} \|u\|_1 \int_0^\infty \exp(-t)t^{-1/2}\,dt = \|u\|_1.\end{aligned}$$

The result follows from the selfadjointness of $(H+1)^{-1/2}$.

3. The operator $H + V$. Let H be the generator of a HCSG and let V be a real function in L^∞. By Proposition 2.4, there exist constants S, E, and m, independent of V such that $(*)$ holds. Thus, $-2S(H+V) \leqslant \ln E + \ln\|\exp(-SV)\|_m$, and for ψ in $\mathcal{D}(H) \cap \mathcal{D}(V)$,

$$\langle (H+V)\psi, \psi\rangle \geqslant -(1/2S)(\ln E + \ln\|\exp(-SV)\|_m)\|\psi\|^2.$$

Now let V be any real function such that $\exp(-SV)$ is in L_m and set

$$V_n = \begin{cases} V, & |V| \leqslant n, \\ n, & |V| > n. \end{cases}$$

Since all the V_n are in L^∞,

$$\langle (H+V_n)\psi, \psi\rangle \geqslant -\frac{1}{2S}(\ln E + \ln\|\exp(-SV_n)\|_m)\|\psi\|^2, \qquad \psi \in \mathcal{D}(H) \cap \mathcal{D}(V).$$

As $n \longrightarrow \infty$, $\langle (H+V_n)\psi, \psi\rangle \longrightarrow \langle (H+V)\psi, \psi\rangle$ and $\ln\|\exp(-SV_n)\|_m \longrightarrow$

$\ln \|\exp(-SV)\|_m$, by the Dominated Convergence Theorem [5]. Thus, for real V such that $\exp(-SV) \in L^m$,

$$\int_M V|\psi|^2\, du \leqslant \langle H\psi, \psi\rangle + \frac{1}{2S}(\ln E + \ln \|\exp(-SV)\|_m)\|\psi\|^2.$$

This result can be placed in an Orlicz space context. We list briefly the facts we use concerning Orlicz spaces. For proofs and more extensive coverage, see [1], [3], [6], [8], [13], and [14].

If $\Phi(x)$ is a Young's function (i.e., the integral of a positive convex function on $[0, \infty)$ with $\Phi(0) = 0$), the Orlicz space L_Φ is the collection of all functions satisfying

$$\|u\|_\Phi = \inf\left\{\lambda > 0\colon \int_M \Phi\left(\frac{|u|}{\lambda}\right) d\mu \leqslant 1\right\} < \infty.$$

$\|\cdot\|_\Phi$ is a norm, and equipped with this norm, L_Φ is a Banach space. If the constant 1 in the definition of $\|\cdot\|_\Phi$ is changed to any other positive real, the result is a norm equivalent to $\|\ \|_\Phi$. If Ψ is the Young's complement to Φ, a norm equivalent to $\|\ \|_\Phi$ is

$$|||u|||_\Phi = \sup\left\{\int_M |u| \cdot |v|\, du\right\},$$

where the supremum is taken over all v satisfying $\int_M \Psi(|v|)\, du \leqslant 1$.

If there exist positive constants M and x_0 such that $x > x_0$ implies $\Phi(2x) \leqslant M\Phi(x)$, then Φ is said to satisfy the Δ_2-condition. (This definition uses the fact that M is a probability space. For nonfinite measure spaces a variation is necessary.) If both Φ and its Young's complement Ψ satisfy the Δ_2-condition (as in the case $\Phi(x) = x^p$, $p > 1$), then L_Φ is reflexive with dual space L_Ψ. Interestingly, if L_Φ has the norm $\|\ \|_\Phi$, then the operator norm on L_Ψ is $|||\ |||_\Psi$.

It is usually difficult or impossible to find a closed expression for the Young's complement to a given Φ. However, it may be possible to "perturb" Φ so the Orlicz space generated by the new function is the same as that generated by the original (with perhaps a different but equivalent norm), but the Young's complement of the new function is easily calculated. One such perturbation is:

3.1 PROPOSITION. *If Φ is a Young's function satisfying the Δ_2-condition and Γ is a nonnegative function such that*

(i) $\Phi + \Gamma$ *is a Young's function,*

(ii) $\lim_{x\to\infty} \Gamma(x)/\Phi(x) = 0$,

then $L_\Phi = L_{\Phi+\Gamma}$ (i.e., *the sets of functions are the same and the norms* $\|\cdot\|_\Phi$ *and* $\|\cdot\|_{\Phi+\Gamma}$ *are equivalent*).

PROOF. The proof is computational and follows easily from the definitions.

Using the above, the following results are easily obtained. For $x \geqslant 1$, set $\ln^+ x = \ln x$, and let it be zero otherwise. For $p \geqslant 2, m > 0$, denote the Orlicz space with generator $x^p(\ln^+ x)^m$ by $L^p(\ln^+ L)^m$ and its norm by $\|\cdot\|_{p,m}$. The dual to $L^p(\ln^+ L)^m$ is denoted $L^q/(\ln^+ L)^{qm/p}$, $1/p + 1/q = 1$. This is the space of all functions ψ satisfying $\int_M (|\psi|^q/[1 + \ln^+ |\psi|]^{qm/p})\, du < \infty$. The norm $\|\psi\|_{q,-qm/p}$ is defined by

$$\|\psi\|_{q,-qm/p} = \inf\left\{\lambda > 0\colon \int_M \frac{(|\psi|/\lambda)^q}{[\xi + \ln^+(|\psi|/\lambda)]^{qm/p}}\, d\mu \leqslant 1\right\}$$

where $\xi = \max(1, qm/p)$. ξ is used purely to simplify calculations. Also, the dual to $L(\ln^+ L)$ is the set of those functions V satisfying, for some fixed $\alpha > 1$,

$$\|V\|_{1,-1} = \inf\left\{\lambda > 0\colon \int_M \exp(\alpha|V|/\lambda)\, d\mu \geqslant 1\right\} < \infty.$$

Changing α produces a different but equivalent norm.

3.2 THEOREM. *$(H + 1)^{-1/2}$ is a bounded operator from L^2 to $L^2 \ln^+ L$.*

PROOF. By the remarks at the beginning of this section, if u is in $\mathcal{D}(H) \cap \mathcal{D}(V)$, and V is such that $\int_M \exp(mS|V|)\, d\mu \leqslant 1$, then

$$\int V|u|^2\, d\mu \leqslant \langle Hu, u\rangle + ((1/2S)\ln E)\|u\|^2.$$

Thus, by the definition of $L \ln^+ L$, u^2 is in $L \ln^+ L$ with

$$|||u^2|||_{1,1} \leqslant \langle Hu, u\rangle + ((1/2S)\ln E)\, \|u\|^2.$$

Since $|||\cdot|||_{1,1}$ is equivalent to $\|\cdot\|_{1,1}$, we write by an abuse of notation,

$$\|u^2\|_{1,1} \leqslant \langle Hu, u\rangle + ((1/2S)\ln E)\|u\|^2.$$

If $N \geqslant \|u^2\|_{1,1}$, then

$$\int_M \left|\frac{u^2}{N}\right| \ln^+ \left|\frac{u^2}{N}\right| d\mu \leqslant 1, \quad \text{or,} \quad \int_M \left|\frac{u}{\sqrt{N}}\right|^2 \ln^+ \left|\frac{u}{\sqrt{N}}\right| d\mu \leqslant \frac{1}{2}.$$

If the definition of $\|\ \|_{2,1}$ is

$$\|\psi\|_{2,1} = \inf\left\{\lambda > 0\colon \int_M \left|\frac{\psi^2}{\lambda}\right| \ln^+ \left|\frac{v}{\lambda}\right| d\mu \leqslant \frac{1}{2}\right\},$$

it then follows that $u^2 \in L\ln^+L$ implies $u \in L^2 \ln^+L$, and $\|u\|_{2,1}^2 \leq K\|u^2\|_{1,1}$. K is a constant arising from the substitute of equivalent norms.

Thus, for u in $\mathcal{D}(H) \cap \mathcal{D}(V)$,

$$\|u\|_{2,1}^2 \leq \langle Hu, u\rangle + ((1/2S)\ln E)\|u\|^2.$$

An examination of the proof of Proposition 2.4 shows that E can be chosen so $(1/2S)\ln E = 1$. We assume that this has been done. Since u in $\mathcal{D}(H) \cap \mathcal{D}(V)$ is dense in the domain of $H + 1$, we can say $u \in \mathcal{D}(H + 1)$ implies $u \in L^2 \ln^+L$, and $\|u\|_{2,1}^2 \leq \langle (H + 1)u, u\rangle$. Since $\mathcal{D}(H + 1)$ is dense in $\mathcal{D}((H + 1)^{1/2})$, it follows that for $v \in \mathcal{D}((H + 1)^{1/2})$, $\|v\|_{2,1}^2 \leq \|(H + 1)^{1/2}v\|^2$. Setting $f = (H + 1)^{1/2}v$ completes the proof.

We note that part of the method used in the preceding proof yields the following corollary. For any selfadjoint operator A on a Hilbert space, we write $\mathcal{Q}(A) = \mathcal{D}((A^*A)^{1/4})$ for its form domain.

3.3 COROLLARY. *Let H be a nonnegative selfadjoint operator in $L^2(\mu)$. Assume there is a strictly positive constant α such that for real function V satisfying $\int_M \exp(\alpha|V|)\,d\mu < \infty$ we have $\mathcal{Q}(V) \supset \mathcal{Q}(H)$. Then $(H + L)^{-1/2}$ is a bounded operator from L^2 to $L^2 \ln^+L$.*

PROOF. If $\int_M \exp(\alpha|V|)\,d\mu < \infty$, then by assumption, for any f in the domain of $(H + 1)^{1/2}$, $\int_M \||V|^{1/2}f|^2\,d\mu < \infty$. That is, $\int_M |V||f|^2\,d\mu < \infty$. Thus $|f|^2$ is in $L\ln^+L$, and f is in $L^2 \ln^+L$. Hence the range of $(H + 1)^{-1/2}$ is contained in $L^2 \ln^+L$. Since $(H + 1)^{-1/2}$ is bounded from L^2 to L^2, one sees that it is closed as an operator from L^2 to $L^2 \ln^+L$ and therefore is bounded from L^2 to $L^2 \ln^+L$ by the Closed Graph Theorem.

3.4 REMARK. The hypothesis $\mathcal{Q}(V) \supset \mathcal{Q}(H)$ is natural in the context of questions concerning the semiboundedness of $H + V$ (form sum) which underlie Theorem 3.2.

4. Interpolation and extension. To study the action of $(H + 1)^{-1/2}$ and its iterations on L^p spaces for $p \geq 2$, we rely heavily on the techniques in Calderón's paper [2]. We state briefly those results that we will use.

If B_0 and B_1 are complex Banach spaces continuously embedded in a TVS V (such a pair of spaces is called an interpolation pair), then there is a Banach space denoted B_s or $[B_0, B_1]_s$ interpolated between B_0 and B_1 in the following sense:

4.1 PROPOSITION. *Let (B_0, B_1) and (C_0, C_1) be two interpolation pairs and let $L\colon B_0 + B_1 \longrightarrow C_0 + C_1$ be a linear mapping such that $x \in B_i$ implies $Lx \in C_i$ and $\|Lx\|_{C_i} \leq M_i\|x\|_{B_i}$ $(i = 0, 1)$. Then $x \in B_s$ implies $Lx \in C_s$ and $\|Lx\|_{C_s} \leq M_0^{1-s}M_1^s\|x\|_{B_s}$.*

PROOF. See [2].

Unfortunately, it is rarely easy to identify the intermediate spaces, so the equivalence of this method and another, where the identification is direct, is shown. Let X be a Banach lattice (i.e., a Banach space of functions measurable over some measure space M with the property that $f \in X$, $|g(x)| \leqslant |f(x)|$ a.e. implies $g \in X$ and $\|g\|_X \leqslant \|f\|_X$). Let $\phi(x, t)$ be, for each $x \in X$, a concave increasing function on $[0, \infty)$ vanishing at 0. Then the collection $\phi(X)$ of all measurable functions $g(x)$ satisfying $|g(x)| \leqslant \lambda\phi(x, |f(x)|)$ a.e. for some $f \in X$, $\|f\|_X \leqslant 1$ and some $\lambda > 0$ is itself a Banach lattice, with norm the infimum of all λ for which such an inequality holds. In particular, if $X = L^1(M)$ and $\phi(x, t) = \phi(t)$ is the inverse of some Young's function Φ, then $\phi(X)$ is the Orlicz space L_Φ.

Also, if X_0 and X_1 are Banach lattices and $0 < s < 1$, then the collection $X_0^{1-s}X_1^s$ of measurable functions $h(x)$ satisfying $|h(x)| \leqslant \lambda |f_0(x)|^{1-s}|f_1(x)|^s$ a.e. for some $f_i \in X_i$, $\|f_i\|_i \leqslant 1$ $(i = 0, 1)$ and some $\lambda > 0$ is again a Banach lattice with norm defined as in $\phi(X)$. If $\phi_0(x, t)$ and $\phi_1(x, t)$ are as above and X is a Banach lattice, then $(\phi_0^{1-s}\phi_1^s)(X)$ is the same as $(\phi_0(X))^{1-s}(\phi_1(X))^s$.

Finally, if B is a Banach space and X a Banach lattice on M, then $X(B)$, the class of all B-valued measurable functions $f(x)$ with $\|f(x)\|_B \in X$ is a Banach space with norm $\|f\|_{X(B)} = \|(\|f(x)\|_B)\|_X$.

4.2 PROPOSITION. *Let (B_0, B_1) be an interpolation pair and X_0, X_1 be two Banach lattices. Then $X_0(B_0)$ and $X_1(B_1)$ are continuously embedded in $(X_0 + X_1)(B_0 + B_1)$. If $X = X_0^{1-s}X_1^s$ and $B = [B_0, B_1]s$, then*

$$[X_0(B_0), X_1(B_1)]_s = X(B),$$

provided X has the dominated convergence property (i.e., $f \in X$, $|f_n| \leqslant |f|$ a.e. and $f_n \longrightarrow 0$ a.e. implies $\|f_n\|_x \longrightarrow 0$).

PROOF. See [2].

All the spaces dealt with here have the dominated convergence property.

4.3 THEOREM. $(H + 1)^{-1/2}$ *is bounded from L^p to $L^p \ln^+ L$ for $2 \leqslant p < \infty$.*

PROOF. It is enough to show that L^p and $L^p \ln^+ L$ are spaces of the form $[X_0(B_0), X_1(B_1)]_s$ and the results will follow by Propositions 2.5, 3.2, 4.1, and 4.2.

Let $B_0 = B_1 = C$, the complex numbers; then $B_s = C$, $0 < s < 1$. Let $\phi_1(t)$ be the function inverse to $L^2 \ln^+ L$, so $(\phi_1(L^1))(B_0)$ is the Orlicz space $L^2 \ln^+ L$. Let $\phi_\infty(0) = 0$ and $\phi_\infty(t) = 1$, $t > 0$. Then $(\phi_\infty(L^1))(B_1) = L^\infty$. The function $\phi_s(t) = \phi_1^{1-s}(t)\phi_\infty^s(t)$ is inverse to $(1/(1-s))t^{2/(1-s)}\ln^+ L$, which generates the Orlicz space $L^{2/(1-s)}\ln^+ L$. Thus,

$$[(\phi_1(L^1))(B_0), (\phi_\infty(L^1))(B_1)]_s = (\phi_s(L_1))(B_s) = L^{2/(1-s)}\ln^+ L.$$

In the same manner the space so interpolated between L^2 and L^∞ is $L^{2/(1-s)}$ and the result follows.

4.4 THEOREM. *$(H + 1)^{-1/2}$ is bounded from $L^q/(\ln^+ L)^{q/p}$ to L^q for $1 < q \leqslant 2$, $1/p + 1/q = 1$.*

PROOF. $(H + 1)^{-1/2}$ is selfadjoint, L^q is the dual of L^p and $L^q/(\ln^+ L)^{q/p}$ is the dual of $L^p(\ln^+ L)$ in the pairing $(f, g) \longrightarrow \int_M f\bar{g}\, d\mu$.

To study the action of iterations of $(H + 1)^{-1/2}$ on L^p, we first examine its behavior on spaces of the form $L^2(\ln^+ L)^m$, m a positive integer, and then interpolate as before. For convenience we denote $(H + 1)^{-1/2}$ by T.

4.5 LEMMA. *Let j be a positive integer and let h be in $L^2/(\ln^+ L)^{2j}$. Then there exists g in L^2 such that $h = g(\ln |g|)^j$ and $\|g\|_2 \leqslant K\|h\|_{2,-2j}$, where K is a constant depending only on j.*

PROOF. Define g by the equation $h = g(\ln |g|)^j$, choosing, for a given value of $h(x)$, the larger of the two possible values for g if ambiguity exists. To show the L^2-norm of g is dominated by the $L^2/(\ln^+ L)^{2j}$-norm of h, notice first that $\Psi(x) = x^2/(2j + \ln^+ x)^{2j}$ has the Δ_2-property with $M = 4$. Choose m so large that $2^{m-1} \leqslant \|h\|_{2,-2j} < 2^m$. Then, since Ψ is increasing, $\int_M \Psi(|h|/2^m)\, d\mu \leqslant 1$, and thus by repeated application of the Δ_2-property, $\int_M \Psi(|h|)\, du \leqslant 4^m$. Thus,

$$\int_M \frac{|g|^2(\ln |g|)^{2j}}{[2j + (j + 1)\ln |g|]^{2j}}\, d\mu \leqslant \int_M \frac{|g|^2(\ln\ |g|)^{2j}}{[2j + \ln^+(g(\ln |g|)^j)]^{2j}}\, d\mu \leqslant 4^m.$$

The first integral in the above is less than a constant times $\int_M |g|^2\, d\mu$, the constant depending only on j. Therefore,

$$\|g\|_2 \leqslant k(j)2^m = 2k(j)2^{m-1} \leqslant 2k(j)\|h\|_{2,-2j}.$$

We now prove that T is bounded as an operator from $L^2(\ln^+ L)^n$ to $L^2(\ln^+ L)^{n+1}$, n a positive integer. The proof is in two parts.

4.6 THEOREM. *Let $n \geqslant 1$ be an integer. Then T is bounded as an operator from $L^2(\ln^+ L)^{2n-1}$ to $L^2(\ln^+ L)^{2n}$.*

PROOF. We proceed by induction on n. Let f be a positive simple function in $L^2(\ln^+ L)$ and g a positive simple function in L^2. Consider the complex function

$$B_z(f, g) = \int_M T(f^{2-2z})(g^{2z})\, d\mu.$$

$B_z(f, g)$ is analytic in $0 < \mathrm{Re}(z) < 1$ and continuous in $0 \leqslant \mathrm{Re}(z) \leqslant 1$. Furthermore, $|B_{it}(f, g)| \leqslant \int T(f^2)\, d\mu$. (We use here the fact that T is positivity-preserving, which follows from the hypotheses that $\exp(-tH)$ is, and from the weak-integral representation of $(H + 1)^{-1/2}$.) Since f is in $L^2(\ln^+ L)$, f^2 is in L^1 and $\|f^2\|_1 \leqslant \|f\|_{2,1}^2(e^2 + 1)$. Since T is bounded on L^1, $|B_{it}(f, g)| \leqslant K\|f\|_{2,1}^2$, where K is a constant. Similarly, $|B_{1+it}(f, g)| \leqslant L\|g\|^2$. Thus, by the Three Lines Theorem [8],

$$|B_{s+it}(f, g)| \leqslant K^{1-s}L^s\|f\|_{2,1}^{2(1-s)}\|g\|^{2s}, \qquad 0 < s < 1.$$

Assuming w.l.o.g. that both $\|f\|_{2,1}$ and $\|g\|$ are greater than 1, we can write $|B_z(f, g)| \leqslant M\|f\|_{2,1}^2\|g\|^2$ for $0 < \mathrm{Re}(z) < 1$. It follows by the Cauchy estimates [9] that a similar bound, with a different constant, holds for all the derivatives of $B_z(f, g)$ also, in particular $B'_{1/2}(f, g)$.

$$B'_{1/2}(f, g) = -2\int_M T(f \ln f)(g)\, d\mu + 2\int T(f)(g \ln g)\, d\mu,$$

so

$$\left|\int_M T(f)(g \ln g)\, d\mu\right| \leqslant \frac{1}{2}|B'_{1/2}(f, g)| + \int_M T(f \ln f) g\, d\mu.$$

The first term on the RHS of the above inequality is dominated by a constant times $\|f\|_{2,1}^2\|g\|^2$. Since f is in $L^2 \ln L$, $f \ln f$ is in $L^2/(\ln^+ L)$ and it is easy to show that $\|f \ln f\|_{2,-1} \leqslant \|f\|_{2,1}$. Since T is selfadjoint, $T(f \ln f)$ is thus in L^2, and $\|T(f \ln f)\| \leqslant K\|f \ln f\|_{2,-1} \leqslant K\|f\|_{2,1}$. Thus the second term on the RHS is dominated by $N\|f\|_{2,1}\|g\|$, where N is a constant. Thus,

$$\left|\int_M T(f) g \ln g\, d\mu\right| \leqslant \text{constant } \|f\|_{2,1}^2\|g\|^2.$$

Now let h be a simple positive function in $L^2/(\ln^+ L)^2$. By Lemma 4.5, there is a positive function g in L^2 with $h = g \ln g$, and $\|g\| \leqslant K\|h\|_{2,-2}$. Using this h in the above result, we obtain

$$\left|\int_M T(f) h\, d\mu\right| \leqslant \text{constant } \|f\|_{2,1}^2\|h\|_{2,-2}^2,$$

and so by homogeneity,

$$\left|\int_M T(f) h\, d\mu\right| \leqslant \text{constant } \|f\|_{2,1}\|h\|_{2,-2}.$$

This result extends to all simple functions by appealing to the fact that T is positivity-preserving, and to all functions in $L^2(\ln^+ L)$ and $L^2/(\ln^+ L)^2$ by observing that the simple functions are dense in both spaces. Since $L^2/(\ln^+ L)^2$ is the dual of $L^2(\ln^+ L)^2$, it follows that T is bounded from $L^2 \ln^+ L$ to $L^2(\ln^+ L)^2$, and thus the base step of the induction is completed.

Now assume that T is bounded from $L^2(\ln^+ L)^{2j-1}$ to $L^2(\ln^+ L)^{2j}, j = 1, 2, 3, \ldots, n-1$. Let f be a positive simple function in $L^2(\ln^+ L)^{2n-1}$ and let g be a positive simple function in L^2. Consider the nth derivative of $B_z(f, g)$:

$$B_z^{(n)}(f, g) = 2^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_M T(f^{2-2z}(\ln f)^j) g^{2z} (\ln g)^{n-j} \, d\mu.$$

As before, we examine $B_{1/2}^{(n)}(f, g)$. Since f is in $L^2(\ln^+ L)^{2n-1}$, f^2 is in L^1, and as before, the L^1-norm of f^2 is bounded by a power of the $L^2(\ln^+ L)^{2n-1}$-norm of f. Thus, again via the Three Lines Theorem and the Cauchy estimates [9], we have $|B_{1/2}^{(n)}(f, g)| \leqslant$ constant $\|f\|_{2,2n-1}^{\alpha} \|g\|_2^2$, where α is a constant depending only on n. We will show that all terms except the $j = 0$ term on the RHS of the equation for $B_z^{(n)}(f, g)$ are similarly dominated, and thus the $j = 0$ term must be also. Then, setting $h = g(\ln g)^n$, as before, $h \in L^2/(\ln^+ L)^{2n}$ and $\|g\| \leqslant K \|h\|_{2,-2n}$, we will have shown

$$\left| \int T(f) h \, d\mu \right| \leqslant \text{constant } \|f\|_{2,2n-1}^{\alpha} \|h\|_{2,-2n}$$

and as above, this will establish that T is bounded from $L^2(\ln^+ L)^{2n-1}$ to $L^2(\ln^+ L)^{2n}$, again appealing to the positivity-preserving property of T. Now, let $0 < j \leqslant n$. Since f is in $L^2(\ln^+ L)^{2n-1}$, $f(\ln f)^j$ is in $L^2(\ln^+ L)^{2(n-j)-1}$, with $\|f(\ln f)^j\|_{2,2(n-j)-1} \leqslant$ constant $\|f\|_{2,2n-1}^{\gamma}$, where γ is a constant independent of f. By the induction hypothesis,

$$\|T(f(\ln f)^j)\|_{2,2(n-j)} \leqslant \text{constant} \, \|f(\ln f)^j\|_{2,2(n-j)-1}.$$

Also, since g is in L^2, $g(\ln g)^{n-j}$ is in $L^2/(\ln^+ L)^{2(n-j)}$, and $\|g(\ln g)^{n-j}\|_{2,-2(n-j)} \leqslant$ constant $\|g\|_2$. Therefore, since $L^2(\ln^+ L)^{2(n-j)}$ is the dual to $L^2(\ln^+ L)^{2(n-j)}$,

$$\left| \int_M T(f(\ln f)^j) g (\ln g)^{n-j} \, d\mu \right| \leqslant \text{constant } \|f\|_{2,2n-1}^{\gamma} \|g\|_2,$$

and so by the previous remarks, the theorem is proven.

Finally, using an interpolation argument, we show T is bounded as an operator from $L^2(\ln^+ L)^{2n}$ to $L^2(\ln^+ L)^{2n+1}$.

4.7 THEOREM. *T is bounded from $L^2(\ln^+ L)^{2n}$ to $L^2(\ln^+ L)^{2n+1}$ for n a positive integer.*

PROOF. By Theorem 4.6, T is bounded from $L^2(\ln^+ L)^{2n-1}$ to $L^2(\ln^+ L)^{2n}$ and from $L^2(\ln^+ L)^{2n+1}$ to $L^2(\ln^+ L)^{2n+2}$. We proceed as in the proof of Theorem 4.3. Let

$$\phi_0(t) = \sqrt{t}/[\tfrac{1}{2}\ln t]^{2n-1} \quad \text{and} \quad \phi_1(t) = \sqrt{t}/[\tfrac{1}{2}\ln t]^{2n+1}.$$

The inverses of ϕ_0 and ϕ_1 generate the Orlicz spaces $L^2(\ln^+ L)^{2n-1}$ and $L^2(\ln^+ L)^{2n-1}$. Then

$$\phi_s(t) = \phi_0^{1-s}(t)\phi_1^s(t) = \sqrt{t}/[\tfrac{1}{2}\ln t]^{2(n+s)-1}.$$

Setting $s = \tfrac{1}{2}$ gives the function inverse to a generator of $L^2(\ln^+ L)^{2n}$. An almost identical argument shows that the space interpolated between $L^2(\ln^+ L)^{2n}$ and $L^2(\ln^+ L)^{2n+2}$ is $L^2(\ln^+ L)^{2n+1}$. The remainder of the proof is exactly the same as that of Theorem 4.3.

5. Extensions.

5.1 THEOREM. *Let $p \leqslant 2$ and let m be a positive integer. Then T is bounded from $L^p(\ln^+ L)^m$ to $L^p(\ln^+ L)^{m+1}$.*

PROOF. The proof is the same as that of Theorem 4.3.

5.2 THEOREM. *Let $p \geqslant 2$, m a positive integer, and $1/p + 1/q = 1$. Then T is bounded from $L^q/(\ln^+ L)^{q(m+1)/p}$ to $L^q/(\ln^+ L)^{qm/p}$.*

PROOF. The dual of $L^p(\ln^+ L)^m$ is $L^q/(\ln^+ L)^{qm/p}$, that of $L^p(\ln^+ L)^{m+1}$ is $L^q/(\ln^+ L)^{q(m+1)/p}$, and T is selfadjoint.

5.3 COROLLARY. *If k is a nonnegative integer, then T^k is bounded from $L^p(\ln^+ L)^m$ to $L^p(\ln^+ L)^{m+k}$ and from $L^q/(\ln^+ L)^{q(m+k)/p}$ to $L^q/(\ln^+ L)^{qm/p}$, where $p \geqslant 2$, $1/p + 1/q = 1$, and m a positive integer.*

PROOF. The result follows by repeated applications of Theorem 5.1 and Theorem 5.2.

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